It is seen from Fig. 6 that conditions (2.3) can be satisfied not only along the integral curves that define flows with a limit line but, also, along some curves for which continuous solutions exist. Furthermore, unlike in the case of inert gas, a shock front may be generated at coordinate $\xi_s < 0$. Flows with the shock wave reaching the nozzle center do not evidently obtain under real conditions. They correspond to flows in nozzles with wall discontinuities.

The author thanks O. S. Ryzhov for formulating the problem and for valuable discussions in the course of this work.

REFERENCES

1. Napolitano, L. G. and Ryzhov, O. S., On the analogy between nonequilibrium and viscous flows at transonic velocities. (English translation), Pergamon Press, J. USSR Comput. Mat. mat. Phys., Vol. 11, № 5, 1971.
2. Ryzhov, O. S., Shock wave formation in Laval nozzles, PMM Vol. 27, № 2, 1963.
3. Frankl', F. I., On the theory of Laval nozzles. Izv. Akad. Nauk SSSR, Ser. Matem. Vol. 9, № 5, 1945.

Translated by J. J. D.

UDC 536.24.01+517.946

ON PROPAGATION OF HEAT IN ONE-DIMENSIONAL DISPERSE MEDIA

PMM Vol. 39, № 6, 1975, pp. 1073-1081

V. G. MARKOV and O. A. OLEINIK
(Moscow)

(Received February 10, 1975)

It is shown that solutions of the first boundary value problem for second order linear parabolic equation with two independent variables reduce in region ω with weak convergence of its coefficients in $L_2(\omega)$ to the solution of the first boundary value problem for some limit equation. This means that solution of the "microscopic" problem of heat propagation in one-dimensional disperse medium can be approximated by the solution of the "macroscopic" problem.

The basic problem of the theory of disperse media consists of the determination of macroscopic properties of these by the known properties of their constituents and by the macroscopic parameters which depend on the disperse medium structure. A strict mathematical formulation of this problem in a general form has not been so far achieved (see surveys [1, 2]). Statistical methods had been applied to the investigation of properties of disperse media [3 — 5]. Another approach consists in the analysis of equations with discontinuous coefficients that define disperse media at a "microscopic" level with the view to approximating solutions of such equations by functions which satisfy equations whose coefficients are in a certain sense limiting and possess better differential properties than the coefficients of input equations (see [6 — 8]). This problem has not yet been analyzed in a general form. Supplementary restrictions were imposed in the considered cases on the structure of coefficients of input equations, as for example, the condition of periodicity [9, 10] or of other kind [6, 11].

The above mentioned macroscopic parameters that define a disperse medium are usually averaged over a small volume.

In the present paper the microscopic parameters that define a disperse medium in region ω are considered to be terms of sequences weakly convergent in $L_2(\omega)$ to some (generally smooth) functions which are taken as the macroscopic characteristics of a disperse medium. For a one-dimensional medium this condition is sufficient for obtaining a uniform convergence of solutions of a "microscopic" problem to that of a "macroscopic" one. Below we present the proof of this statement for the general parabolic second order equation with a single space variable. The method of [11] is used for this. The first boundary value problem is considered, although a similar investigation can be applied to a number of other boundary conditions and, also, to the Cauchy problem.

1. Let function $u^m(x, t)$, $m = 1, 2, \ldots$ satisfy in region $\omega = \{x, t : 0 < x < l,\ 0 < t < T\}$ the boundary value problem for the parabolic type equations

$$L^m(u) \equiv -p^m(x, t)\, u_t + (a^m(x, t)\, u_x)_x + b^m(x, t)\, u_x + c^m(x, t)\, u = f^m(x, t) \qquad (1.1)$$

$$u(x, 0) = u_0^m(x), \quad u(0, t) = u_1^m(t), \quad u(l, t) = u_2^m(t) \qquad (1.2)$$

It is proved that on some specific assumptions $u^m(x, t)$ then to the limit $u(x, t)$ when $m \to \infty$, if p^m, $1/a^m$, b^m/a^m, c^m and f^m weakly converge in $L_2(\omega)$ (see [12]) for $m \to \infty$ to functions p, $1/A$, B, c and f, respectively, and functions u_0^m, u_1^m and u_2^m converge in the mean for $m \to \infty$ to functions u_0, u_1 and u_2, respectively. ($L_2(\omega)$ denotes the space of measurable functions $v(x, t)$ in ω for which $\int_\omega v^2 dxdt < \infty$.) The limit function $u(x, t)$ defines the temperature distribution that corresponds to the limit parabolic equation and boundary conditions of the form

$$I(u) \equiv -p(x, t)\, u_t + (A(x, t)\, u_x)_x + B(x, t)\, A(x, t)\, u_x + c(x, t)\, u = f(x, t) \qquad (1.3)$$

$$u(x, 0) = u_0(x), \quad u(0, t) = u_1(t), \quad u(l, t) = u_2(t) \qquad (1.4)$$

We assume that in ω

$$|p^m| + |a^m| + |b^m| + |c^m| + |f^m| \leqslant M \qquad (1.5)$$

$$p^m \geqslant \alpha_0 > 0, \qquad a^m \geqslant \alpha_1 > 0$$

$$\left| \int_0^x \left(\frac{1}{a^m(s, t)} \right)_t ds \right| \leqslant M, \qquad |p_t^m| \leqslant M$$

$$|u_0^m(x)| + |u_1^m(t)| + |u_2^m(t)| + \left| \frac{du_1^m}{dt} \right| + \left| \frac{du_2^m}{dt} \right| \leqslant M \qquad (1.6)$$

where constants M, α_0 and α_1 are independent of m.

In problems of the theory of disperse media the case in which coefficients of Eq. (1.1) and functions $f^m(x, t)$ are only piecewise continuous and piecewise smooth are of considerable interest. Here the problem (1.1), (1.2) is considered for arbitrary bounded measurable functions p^m, a^m, b^m, c^m and f^m, that satisfy conditions (1.5); hence any discontinuities that are interesting from the physical point of view are permitted for these functions. It is, therefore, necessary to consider in this connection the generalized solutions $u^m(x, t)$ of problem (1. 1), (1. 2).

First, we consider the case, when the coefficients of Eq. (1. 1), function f^m and functions $u_0{}^m$, $u_1{}^m$ and $u_2{}^m$ satisfy the conditions of smoothness and of matching at points $(0, 0)$ and $(0, l)$ for which there exists a solution of problem (1. 1), (1. 2) whose derivatives appearing in Eq. (1. 1) are continuous in $\overline{\omega}$ ($\overline{\omega}$ denotes closing of the set ω) (see, e, g, [13] and Sect. 3 in [14]). Then, using the obtained results, we consider the case of discontinuous coefficients in Eq. (1. 1) which is of the greatest interest in the theory of disperse media.

2. Let $p(x, t)$, $p_t(x, t)$, $A(x, t)$, $B(x, t)$, $c(x, t)$ and $f(x, t)$ be bounded measurable functions in ω; $u_0(x)$ be a bounded measurable function along segment $[0, l]$; $u_1(t)$ and $u_2(t)$ be bounded and continuous for $0 < t < T$, $p \geqslant \alpha_0 > 0$ and $A \geqslant \alpha_2 > 0$ with α_0 and α_2 constant.

Definition. Function $u(x, t)$ which is bounded in ω and continuous in $\overline{\omega}$ for $t > 0$ is called the generalized solution of problem (1. 3), (1. 4), if $u_x \in L_2(\omega)$, $u(0, t) = u_1(t)$ and $u(l, t) = u_2(t)$ for $t > 0$, and if for any infinitely differentiable in $\overline{\omega}$ function $\varphi(x, t)$ such that $\varphi(x, T) = 0$, $\varphi(0, t) = 0$ and $\varphi(l, t) = 0$ the integral identity

$$\int_\omega [(p\varphi)_t u - A u_x \varphi_x + BA u_x \varphi + cu\varphi - f\varphi]\, dx\, dt + \int_0^l p(x, 0)\, \varphi(x, 0)\, u_0\, dx = 0 \quad (2.1)$$

is satisfied.

Theorem 1. If the generalized derivatives A_t and $(BA)_x$ are bounded in ω, then the generalized solution $u(x, t)$ of problem (1. 3), (1. 4) is unique.

Proof. Let us assume the existence of two generalized solutions $u_1(x, t)$ and $u_2(x, t)$ of problem (1. 3), (1. 4), and prove that in ω $u_1 \equiv u_2$. The remainder $u_1 - u_2 = v$ satisfies the integral identity

$$\int_\omega [(p\varphi)_t v - A v_x \varphi_x + BA v_x \varphi + cv\varphi]\, dx\, dt = 0 \quad (2.2)$$

It can be readily shown by passing to limit that the integral identity (2. 2) is also valid for any function $\varphi(x, t)$ such that $\varphi \in L_2(\omega)$, $\varphi_x \in L_2(\omega)$, $\varphi_t \in L_2(\omega)$, $\varphi(x, t) = 0$, $\varphi(0, t) = 0$ and $\varphi(l, t) = 0$. We substitute into (2. 2) the function defined by equalities

$$\varphi(x, t) = e^{\alpha t}\psi(x, t) \quad 0 \leqslant t \leqslant T_1$$

$$\varphi(x, t) = 0, \quad T_1 \leqslant t \leqslant T; \qquad \psi(x, t) = \int_t^{T_1} v(x, s)\, ds$$

for $\varphi(x, t)$. The positive constants α and T_1 will be chosen later. We have

$$\int_\omega [v p_t \psi - v^2 p + \alpha v p \psi - A v_x \psi_x + BA v_x \psi + cv\psi]\, e^{\alpha t}\, dx\, dt = 0 \quad (2.3)$$

We transform individual terms of equality (2. 3) by integration by parts, and obtain

$$\int_\omega A v_x \psi_x e^{\alpha t}\, dx\, dt = \int_\omega \frac{1}{2} e^{\alpha t} (A_t + \alpha A)\, \psi_x{}^2\, dx\, dt + \int_0^l \frac{1}{2} A(x, 0)\, (\psi_x(x, 0))^2\, dx \quad (2.4)$$

$$\int_{\omega} e^{\alpha t} BAv_x \psi \, dx\, dt = -\int_{\omega} e^{\alpha t} v\, [BA\psi_x + (BA)_x \psi] \cdot dx\, dt \qquad (2.5)$$

Taking into accont formulas (2.4) and (2.5) and using the Cauchy-Buniakowski and the elementary inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$, for the estimates of integrals in equality (2.3), we obtain

$$\left| \int_{\omega} [p_t + \alpha p + c - (BA)_x]\, e^{\alpha t} v \psi \, dx\, dt \right| \leqslant$$

$$(K_1 + \alpha K_2)\, e^{\alpha T_1} \left(\int_{\omega} v^2\, dx\, dt \right)^{1/2} \left(\int_{\omega} \psi^2\, dx\, dt \right)^{1/2} \leqslant$$

$$\left[\varepsilon + \frac{(K_1 + \alpha K_2)^2}{\varepsilon}\, e^{2\alpha T_1} T_1^2 \right] \int_{\omega} v^2\, dx\, dt$$

$$\left| \int_{\omega} e^{\alpha t} BAv \psi_x \, dx\, dt \right| \leqslant \varepsilon I(v) + \frac{K_3}{\varepsilon} I(\psi_x),$$

$$I(q) = \int_{\omega} q^2 e^{\alpha t}\, dx\, dt$$

where ε is an arbitrary positive constant and constants K_1, K_2 and K_3 are independent of ε, T_1 and α. Taking into account these estimates, from Eq. (2.3) we conclude that

$$\int_{\omega} \left[pv^2 + \frac{1}{2} A\alpha \psi_x^2 + \frac{1}{2} A_t \psi_x^2 \right] e^{\alpha t}\, dx\, dt + \int_0^l \frac{1}{2} A(x, 0)\, \psi_x^2\, dx \leqslant \qquad (2.6)$$

$$\left(2\varepsilon + \frac{(K_1 + \alpha K_2)^2}{\varepsilon} T_1^2 e^{2\alpha T_1} \right) I(v) + \frac{K_3}{\varepsilon} I(\psi_x)$$

We set $\varepsilon = \alpha_0/4$ and select $\alpha \geqslant \alpha^{-1}_2 (\sup \omega \mid A_t \mid + 2\varepsilon^{-1} K_3)$, and T_1 so small that

$$\frac{1}{2} \alpha_0 > (K_1 + \alpha K_2)^2\, \varepsilon^{-1} T_1^2 e^{2\alpha T_1}$$

Then it follows from (3.6) that $I(v) \leqslant 0$, and, consequently, $v \equiv 0$ in ω , when $0 < t \leqslant T_1$. We prove in the same manner that $v \equiv 0$ for $T_1 \leqslant t \leqslant 2T_1, \ldots, kT_1 \leqslant t \leqslant T$, where k is equal to the integral part of T/T_1. The theorem is proved.

Note that the theorem of uniqueness of the generalized solution $u(x, t)$ and that of existence of a smooth solution of problem (1.3), (1.4), proved in [13, 14] imply that when the coefficients of Eq. (1.3), function $f(x, t)$ and the functions in conditions (1.4) are reasonably smooth and satisfy the conditions of merging at points $(0, 0)$ and $(0, l)$, the generalized solution of problem (1.3), (1.4) is a function that has in $\overline{\omega}$ continuous derivatives u_t, u_x and u_{xx}.

3. Let us consider the case of reasonably smooth functions p^m, a^m, b^m, c^m, f^m and u_0^m, u_1^m, u_2^m.

Theorem 2. Let $u^m(x, t)$ be the solution of problem (1.1), (1.2) , whose derivatives u_t^m, u_x^m and u_{xx}^m are continuous in $\overline{\omega}$. We assume that conditions (1.5) and (1.6) are satisfied and that for $m \to \infty$ functions p^m, p_t^m, $1/a^m$, b^m/a^m, c^m and f^m weakly converge in $L_2(\omega)$ to functions p, p_t, $1/A$, B, c and f, respectively; that functions $p^m(x, 0)$ weakly converge in $L_2(0, l)$ to function $p(x, 0)$; $u_0^m(x)$ converges in norm $L_2(0, l)$ to function $u_0(x)$, and that $u_1^m(t)$ and $u_2^m(t)$ converge in norm $L_2\,(0, T)$ to functions $u_1(t)$ and $u_2(t)$, respectively. We assume that A_l and $(BA)_x$ are bounded in ω. Then for $m \to \infty$ solutions $u^m\,(x, t)$ of problem (1. 1), (1.3) uniformly converge in $\omega_\delta = \{x, t : 0 < x < l,\ \delta < t < T\}$ for any $\delta > 0$ to the generalized solution $u\,(x, t)$ of problem (1. 3), (1. 4).

Proof. According to the principle of maximum (see [13])

$$|\,u^m\,(x, t)\,| \leqslant c_1 \tag{3. 1}$$

where constant c_1 is independent of m. We denote by $w^m\,(x, t)$ the function that satisfies in ω the condition

$$(a^m\,(x, t)\,w_x)_x = 0; \quad w\,(0, t) = u_1{}^m\,(t), \quad w\,(l, t) = u_2{}^m\,(t)$$

It is evident that

$$w^m\,(x, t) = u_1{}^m\,(t) + q^m\,(x, t)\,[q^m\,(l, t)]^{-1}\,[u_2{}^m\,(t) - u_1{}^m\,(t)]$$

$$q^m\,(x, t) = \int_0^x [a^m\,(s, t)]^{-1}\,ds$$

Functions w^m, $w_t{}^m$ and $w_x{}^m$ are by virtue of conditions (1. 5) and (1. 6) obviously uniformly bounded in ω with respect to m.

To estimate $u_x{}^m$ in norm $L_2\,(\omega)$ we consider functions $v^m = u^m - w^m$. Obviously $v^m\,(0, t) = 0$ and $v^m\,(l, t) = 0$. We multiply Eq. (1. 1) by v^m and integrate it over region ω. Transforming individual terms of the obtained equality by integration by parts, we obtain

$$\int_\omega p^m u_t{}^m v^m\,dx\,dt = \int_\omega \Big[-\frac{1}{2}\,p_t{}^m\,(u^m)^2 + (p^m w^m)_t \cdot u^m\Big]\,dx\,dt + \tag{3. 2}$$
$$\int_0^l \Big[\frac{1}{2}\,p^m\,(x, t)\,(u^m\,(x, t))^2 - \frac{1}{2}\,p^m\,(x, 0)\,(u_0{}^m\,(x))^2\Big]\,dx -$$
$$\int_0^l [p^m\,(x, T)\,u^m\,(x, T)\,w^m\,(x, T) - p^m\,(x, 0)\,u_0{}^m\,(x)\,w^m\,(x, 0)]\,dx$$

By virtue of estimate (3. 1) and assumptions (1. 3) and (1. 6) about functions p^m, a^m, $u_0{}^m$, $u_1{}^m$ and $u_2{}^m$ all integrals in the right-hand part of equality (3. 2) are bounded by a constant independent of m. Furthermore

$$\int_\omega (a^m u_x{}^m)_x\,v^m\,dx\,dt = \int_\omega [-a^m\,(u_x{}^m)^2 + a^m u_x{}^m w_x{}^m]\,dx\,dt$$

Thus we obtain

$$\int_\omega a^m\,(u_x{}^m)^2\,dx\,dt - \int_\omega [a^m u_x w_x{}^m + b^m u_x{}^m w^m]\,dx\,dt = B^m$$

where B^m are bounded by a constant independent of m. Using the Cauchy-Buniakowski inequality and the elementary inequality $2ab \leqslant \varepsilon a^2 + \varepsilon^{-1} b^2$, for the estimate of the second integral in equality (3. 3), we obtain that

$$\int_\omega a^m\,(u_x{}^m)^2\,dx\,dt \leqslant c_2 \tag{3. 4}$$

where constant c_2 is independent of m

Let us consider equation

$$L^m\,(v^m) \equiv L^m\,(u^m) - L^m\,(w^m) = f^m - p^m w_t{}^m + b^m w_x{}^m + c^m w^m \equiv F^m\,(x, t) \tag{3. 5}$$

which is satisfied by function v^m. By virtue of assumptions (1. 5) and (1. 6) functions $F^m(x, t)$ are uniformly bounded with respect to m. We substitute in this equation the independent variables of the form

$$\tau = t, \quad y = \frac{q^m(x, t)}{q^m(l, t)} \equiv \Phi^m(x, t)$$

As the consequence of this substitution of variables region ω becomes $\Omega = \{y, \tau : 0 < y < 1, \ 0 < \tau < T\}$ and Eq. (3. 5) becomes

$$-p^m v_\tau^{\ m} - p^m \Phi_\tau^{\ m} v_y^{\ m} + a^m (\Phi_x^{\ m})^2 v_{yy}^{\ m} + b^m \Phi_x^{\ m} v_y^{\ m} + c^m v^m = F^m \tag{3. 6}$$

We introduce the notations

$$\Omega_\delta = \{y, \tau : 0 < y < 1, \ \delta < \tau < T\}, \ \delta = \text{const} > 0$$

$$\| v \|_\gamma^D = \sup_D | v(y, \tau) | + \sup_{\substack{(y_1, \tau_1) \in D \\ (y_2, \tau_2) \in D}} \frac{| v(x_1, \tau_1) - v(y_2, \tau_2) |}{(| \tau_1 - \tau_2 | + | y_1 - y_2 |^2)^{\gamma/2}}$$

where $\gamma = \text{const}$, $0 < \gamma < 1$, and D is a region in space (y, τ). To estimate $v_y^{\ m}$ we apply to Eq. (3. 6) with conditions $v^m(0, t) = 0$ and $v^m(l, t) = 0$ Theorem 3 of [15], in accordance with which

$$\| v^m \|_\gamma^{\Omega_\delta} + \| v_y^{\ m} \|_\gamma^{\Omega_\delta} \leqslant c_3 \tag{3. 7}$$

where constants γ and c_3 are independent of m (but may depend on δ).

It follows from estimate (3. 7) that the set of functions $\{v^m\}$ and $\{q^m(l, t)\, a^m v_x^{\ m}\}$ are uniformly bounded and satisfy Hölder condition with exponent $\gamma/2$ with respect to variables y and τ in the region Ω_δ with the Hölder constant independent of m. Since the derivatives of $q^m(l, t)$ and $\Phi^m(x, t)$ with respect to x and t are uniformly bounded with respect to m and $q^m(x, t) \geqslant \alpha_3 = \text{const} > 0$, hence the sets $\{v^m\}$ and $\{a^m v_x^{\ m}\}$ are uniformly bounded and satisfy the Hölder condition with exponent $\gamma/2$ with respect to variables x and t in any region ω_δ with the Hölder constant independent of m. Sets $\{u^m\}$ and $\{a^m u^m\}$ have the same property, because $v^m = u^m - w^m$ and w^m are uniformly bound with respect to m, while the derivatives of $a^m w_x^{\ m}$ with respect to t and x are bounded with respect to m in ω. Consequently the sets of functions $\{u^m\}$ and $\{a^m u_x^{\ m}\}$ are, according to the Arzelà theorem, compact in the sense of uniform convergence in any region ω_δ with $\delta = \text{const} > 0$.

Using the diagonal process we eliminate the sequence of numbers m_k such that for $m_k \to \infty$ functions u^{m_k} and $a^{m_k} u_x^{\ m_k}$ converge in ω to functions $u(x, t)$ and $V(x, t)$, and the convergence is uniform in any region ω_δ and the sequence $u_x^{\ m_k}$ weakly converges in $L_2(\omega)$ to $u_x(x, t)$.

Note that

$$u_x^{\ m} = \frac{1}{a^m} (a^m u_x^{\ m}) \tag{3. 8}$$

Since for $m_k \to \infty$ $a^{m_k} u_x^{\ m_k}$ uniformly converge in ω_δ to V and $a^m u_x^{\ m}$ are uniformly bounded in norm $L_2(\omega)$ with respect to m, while $1/a^m(x, t)$ weakly converges in $L_2(\omega)$ to $1/A(x, t)$, hence, passing to limit in equality (3. 8) with respect to the chosen sequence m_k, we obtain the equality of limits $u_x = V/A$ which are weak in $L_2(\omega)$. Since $u_x \in L_2(\omega)$, hence $V = A u_x \in L_2(\omega)$. Multiplying Eq. (1. 1) by the infinitely differentiable function $\varphi(x, t)$ such that $\varphi(x, T) = 0$, $\varphi(0, t) = 0$ and $\varphi(l, t) = 0$, integrating the obtained equality over region ω, and transforming its terms by integration by parts, we obtain that the integral identity

$$\int\limits_{\omega} [(p^m\varphi)_t - a^m u_x^m \varphi_x + \frac{b^m}{a^m} a^m u_x^m \varphi + c^m u^m \varphi - f^m \varphi]\, dx\, dt + \int\limits_0^l p^m(x, 0)\, u_0^m(x)\, \varphi(x, 0)\, dx = 0 \tag{3.9}$$

satisfies $u^m(x, t)$.

We now pass to limit in the integral identity (3.9) with respect to the previously selected sequence m_k. In doing so we must take into account

$$\left| \int\limits_{\omega} \left[a^m u_x^m \frac{b^m}{a^m} \varphi - A u_x B\varphi\right] dx\, dt \right| \leqslant \left| \int\limits_{\omega_\delta} \frac{b^m}{a^m} (a^m u_x^m - V)\, \varphi dx\, dt \right| + \left| \int\limits_{\omega_\delta} V\varphi \left(\frac{b^m}{a^m} - B\right) dx\, dt \right| + \sqrt{\delta} \left[\int\limits_{\omega \setminus \omega_\delta} (b^m u_x^m \varphi - VB\varphi)^2 dx\, dt \right]^{1/2} \tag{3.10}$$

It will be seen that the left-hand part of inequality (3.10) tends to zero when $m_k \to \infty$, because by virtue of assumptions (1.5) and estimate (3.4) the last integral in its right-hand part does not exceed $\sqrt{\delta}\, K_4$, where K_4 is independent of δ, the first integral in the right-hand part tends to zero for fixed δ and $m_k \to \infty$ owing to the uniform convergence of $a^m u_x^m$ in ω_δ to V, and the second integral tends to zero for fixed δ owing to the weak convergence in $L_2(\omega)$ of function b^m/a^m to B.

The proof that for $m_k \to \infty$

$$\int\limits_{\omega} a^m u_x^m \varphi_x dx\, dt \to \int\limits_{\omega} V\varphi_x dx\, dt = \int\limits_{\omega} A u_x \varphi_x dx\, dt$$

is similar.

We thus obtain that the limit function $u(x, t)$ satisfies the integral identity (2.1). Furthermore, since $u(x, t)$ is bounded in ω and continuous in $\overline{\omega}$ for $t > 0$, hence

$$u(0, t) = u_1(t), \quad u(l, t) = u_2(t), \quad u_x \in L_2(\omega),$$

and by definition $u(x, t)$ is the general solution of problem (1.3), (1.4). In accordance with Theorem 1 the solution of problem (1.1), (1.4) is unique. Hence the complete sequence $u^m(x, t)$ converges for $m \to \infty$ to function $u(x, t)$, and the convergence in ω_δ is uniform for $\delta = \text{const} > 0$. The theorem is proved.

4. Let us consider the case of discontinuous coefficients and functions f^m in Eq. (1.1). For this it is necessary to examine the generalized solutions of problem (1.1), (1.2). We assume that functions p^m, a^m, b^m, c^m and f^m are measurable in ω and that $u_0^m(x)$, $u_1^m(t)$ and $u_2^m(t)$ are measurable along segments $[0, l]$ and $[0, T]$, respectively, and that conditions (1.5) and (1.6) are satisfied.

The generalized solution of problem (1.1), (1.2) is taken to be the function $u^m(x, t)$ bounded in ω and continuous in $\overline{\omega}$ for $t > 0$ such that $u^m(0, t) = u_1^m(t)$, $u^m(l, t) = u_2^m(t)$ and u_x^m belong to $L_2(\omega)$, and for any infinitely differentiable function $\varphi(x, t)$ with conditions $\varphi(x, T) = 0$, $\varphi(0, t) = 0$ and $\varphi(l, t) = 0$ the integral identity (3.9) is satisfied.

It is proved below that the generalized solution $u^m(x, t)$ of problem (1.1), (1.2) exists when conditions (1.5) and (1.6) are satisfied. If it is assumed that derivatives a_t^m and b_x^m exist and are bounded in ω, then it follows from Theorem 1 that the generalized solution $u^m(x, t)$ of problem (1.1), (1.2) is unique. This restriction is subsequently disregarded.

Theorem 3. Let $u^m(x, t)$ be the generalized solution of problem (1.1), (1.2) and let the assumptions formulated in Theorem 2 with respect to the coefficients of Eq. (1.1) and to functions f^m, u_0^m, u_1^m, u_2^m, A_t and $(BA)_x$ be satisfied. Then for $m \to \infty$ the generalized solution $u^m(x, t)$ of problem (1.1), (1.2) converges in ω, and the convergence is uniform in ω_δ for $\delta = \text{const} > 0$, to the generalized solution $u(x, t)$ of problem (1.3), (1.4).

Proof. We approximate coefficients p^m, a^m, b^m, c^m and function f^m by infinitely differentiable in ω functions $p^{m,n}$, $a^{m,n}$, $b^{m,n}$, $c^{m,n}$ and $f^{m,n}$ with $n=1, 2, \ldots$, such that for $n \to \infty$ these functions converge in $L_2(\omega)$ to functions p^m, a^m, b^m, c^m and f^m, with $p^{m,n}(x, 0)$ converging in $L_2(0, l)$ to $p^m(x, 0)$ Then we approximate functions u_0^m, u_1^m and u_2^m by infinitely differentiable functions $u_0^{m,n}$, $u_1^{m,n}$ and $u_2^{m,n}$ which for $n \to \infty$ converge in $L_2(0, l)$ and $L_2(0, T)$, respectively, to functions u_0^m, u_1^m and u_2^m. The above approximations are chosen so that for any m and n conditions (1.5) and (1.6) with constants M, α_0 and α_1 independent of m and n are satisfied by $p^{m,n}$, $a^{m,n}$, $b^{m,n}$, $c^{m,n}$, $f^{m,n}$, $u_0^{m,n}$, $u_1^{m,n}$ and $u_2^{m,n}$. It is also assumed that all new coefficients $p^{m,n}$, $a^{m,n}$, $b^{m,n}$ $c^{m,n}$ and functions $f^{m,n}$, $u_0^{m,n}$, $u_1^{m,n}$ and $u_2^{m,n}$ satisfy for any n, the conditions of merging at points $(0, 0)$ and $(0, l)$, which ensure the existence in ω of solution $u^{m,n}(x, t)$ of the boundary value problem

$$
\begin{gathered}
-p^{m,n} u_t + (a^{m,n} u_x)_x + b^{m,n} u_x + c^{m,n} u = f^{m,n} \\
u(x, 0) = u_0^{m,n}, \quad u(0, t) = u_1^{m,n}, \quad u(l, t) = u_2^{m,n}
\end{gathered}
$$

whose derivatives u_t, u_x and u_{xx} are continuous in $\bar{\omega}$.

It is seen that estimates (3.1), (3.4) and (3.7) are valid for functions $u^{m,n}(x, t)$ with constants c_1, c_2 and c_3 independent of m and n. Solutions $u^{m,n}(x, t)$ satisfy the integral identity

$$
\int_\omega [(p^{m,n}\varphi)_t u^{m,n} - a^{m,n} u_x^{m,n} \varphi_x + b^{m,n} u_x^{m,n} \varphi_x + c^{m,n} u^{m,n} \varphi - f^{m,n} \varphi]\, dx\, dt + \int_0^l p^{m,n}(x, 0)\, u_0^{m,n}(x)\, \varphi(x, 0)\, dx = 0 \tag{4.1}
$$

for any infinitely differentiable function $\varphi(x, t)$ such that $\varphi(x, T) = 0$, $\varphi(0, t) = 0$ and $\varphi(l, t) = 0$.

It follows from estimates (3.1), (3.4) and (3.7) that for a fixed m it is possible to select such sequence $n_k \to \infty$ that $u^{m,n_k} \to u^m$ in ω and the convergence is uniform in ω_δ with $\delta = \text{const} > 0$, while $u_x^{m,n_k} \to u_x^m$ weakly converges in $L_2(\omega)$. Taking into account that $p^{m,n}$, $a^{m,n}$, $b^{m,n}$, $c^{m,n}$ and $f^{m,n}$ converge in norm $L_2(\omega)$ and $p^{m,n}(x, 0)$ and $u_0^{m,n}(x)$ converge in norm $L_2(0, l)$ for $n \to \infty$, and passing to limit in the integral identity (4.1) for $n_k \to \infty$, we obtain that the limit function $u^m(x, t)$ is the generalized solution of problem (1.1), (1.2). By passing to limit we find that estimates (3.1) and (3.4) are valid for functions $u^m(x, t)$ and that the sets $\{u^m(x, t)\}$ and $\{a^m(x, t)\, u^m(x, t)\}$ are uniformly bounded and equicontinuous in ω_δ for $\delta = \text{const} > 0$. Hence a sequence can be found such that u^{m_k} converges to $u(x, t)$ in ω and uniformly in ω_δ, $u_x^{m_k}$ weakly converges in $L_2(\omega)$ to u_x, and $a^{m_k} u_x^{m_k}$ uniformly converges in ω_δ to $V = A u_x$

for $m_k \to \infty$.

Passing to limit for $m_k \to \infty$ in the integral identity (3.9) as in the proof of Theorem 2, we conclude that $u(x, t)$ is the generalized solution of problem (1.3), (1.4). Owing to the uniqueness of the generalized solution of this problem, the complete sequence $u^m(x, t)$ converges for $m \to \infty$ to $u(x, t)$. The theorem is proved.

Note that the theorems similar to Theorems 2 and 3 can be proved in the case when conditions of the form

$$u(x, 0) = u_0^m(x), \quad a^m u_x \,|_{x=0} = u_1^m(t), \quad a^m u_x \,|_{x=l} = u_2^m(t)$$

are substituted for (1.2). The first boundary value problem with boundary conditions of the form

$$u \,|_{x=\beta_1(t)} = u_1^m(t), \quad u \,|_{x=\beta_2(t)} = u_2^m(t)$$

or the boundary value problem with boundary conditions of the form

$$a^m u_x \,|_{x=\beta_1(t)} = u_1^m(t), \quad a^m u_x \,|_{x=\beta_2(t)} = u_2^m(t)$$

and initial condition $u(x, 0) = u_0^m(x)$ can be investigated in region $\omega' = \{x, t: \beta_1(t) < x < \beta_2(t),\ 0 < t < T\}$.

The methods used here are entirely applicable for the investigation of the Cauchy problem for Eq. (1.1) in region $\{x, t : -\infty < x < \infty,\ 0 < t < T\}$ with initial condition $u(x, 0) = u_0^m(x)$.

REFERENCES

1. De Vries, D. A., La conductibilité thermique des matériaux granuleux. Bull. Inst. Internat. Froid, Vol. 32. № 6, 1952.
2. Hashin, Z., Assessment of the self-consistent scheme approximation; conductivity of particulate compositions. J. Composite Materials, Vol. 2, № 3, 1968.
3. Brown, W. F., Solid mixture permittivities. J. Chem. Phys., Vol. 23, № 8, 1955.
4. Bolotin, V. V. and Moskalenko, V. N., Macroscopic coefficients of thermal conductivity and diffusion in micro-inhomogeneous solids. PMTF, №6, 1967. (English translation), J. of Appl. Mech. and Tech. Phys., Vol. 8, №6, Faraday Press, New York.
5. Buevich, Iu. A. and Korneev, Iu. A., On heat and mass transfer in a disperse system. PMTF, №4, 1974. (English translation), J. of Appl. Mech. and Tech. Phys., Vol. 15, №4, Plenum Press, New York.
6. Marino, A. and Spagnolo, S., Un tipo di approssimazione dell operatore $\sum_1^n {}_{ij} D_i (a_{ij}(x) D_j)$ con operatori $\sum_1^n D_j(\beta(x) D_j)$. Ann. Scu. Norm. Sup., Pisa, Vol. 23, fsc. 4, Sep. 3, 1969.
7. Spagnolo, S., Sulla convergenza di soluzioni di equazioni paraboliche et ellittiche. Ann. Scuola Norm. Sup., Pisa, Vol. 22, fsc. 4, Sep. 3, 1969.
8. De Giorgi, E. and Spagnolo, S., Sulla convergenza degli integrali dell' energia per operatori ellittichi dell secondo ordine. Bull. Unione Mat. Ital. Vol. 8, № 3, Sep. 4, 1973.
9. Sanchez-Palencia, E., Comportements local et macroscopique d'un type de milieux physiques hétérogènes. Internat. J. Engng. Sci., Vol. 12, 1974.

10. Bakhvalov, N. S., Fundamental characteristics of bodies of periodic structure. Dokl. Akad. Nauk SSSR, Vol. 218, № 5, 1974.
11. Oleinik, O. A., On heat propagation in certain multi-dimensional disperse media. Uspekhi Matem. Nauk, Vol. 30, № 4, 1975.
12. Smirnov, V. I., A Course of Higher Mathematics, Vol. 5. (English translation), Pergamon Press, Book № 10211, 1964. Distributed in the USA by the Addison-Wesley Publishing Co.
13. Il'in, A. M., Kalashnikov, A. S., and Oleinik, O. A., Second order linear equations of the parabolic kind. Uspekhi Matem. Nauk, Vol. 17, №3, 1962.
14. Friedman, A., Partial Differential Equations of Parabolic Type. Englewood Cliffs, NJ, Prentice-Hall Inc., 1964.
15. Kruzhkov, S. N., Nonlinear parabolic equations with two independent variables. Tr. Mosk. Matem. Obshch., Vol. 16, 1967.

Translated by J. J. D.

UDC 539.3

ON A VARIATIONAL PROBLEM WITH UNKNOWN BOUNDARIES AND THE DETERMINATION OF OPTIMAL SHAPES OF ELASTIC BODIES

PMM Vol. 39, № 6, 1975, pp. 1082-1092

N. V. BANICHUK
(Moscow)

(Received March 21, 1975)

The optimization problem is considered for a partial differential equation of elliptic type. The boundary of the domain in which the equation is given emerges as the control function and is to be determined from the condition of the extremum of the integral of the solution of the boundary value problem. Seeking the extremals is reduced to solving a variational problem without differential constraints. Necessary conditions for optimality are obtained, and shapes of elastic bars possessing the maximum stiffness under torsion are found with their aid.

1. Formulation of the optimization problem and elimination of the differential constraint. We consider a boundary value problem for the partial differential equation

$$(a\varphi_x - c\varphi_y)_x + (b\varphi_y - c\varphi_x)_y + m = 0 \quad (x, y) \in D \tag{1.1}$$

$$\varphi = 0 \quad (x, y) \in \Gamma \tag{1.2}$$

The coefficients a, b, c of (1.1) are assumed given functions of the variables x, y, and $m > 0$ is a given constant, Γ is the boundary of a simply connected domain D.

Let us formulate the following optimization problem. Determine the smooth closed line Γ satisfying the isoperimetric condition of the constant area of the domain D

$$\iint_D dx\,dy = S \tag{1.3}$$

and such that a maximum of the integral functional